MATHEMATICS OF COMPUTATION
VOLUME 60, NUMBER 201
JANUARY 1993, PAGES 303-316

A QUADRATURE FORMULA INVOLVING ZEROS OF BESSEL FUNCTIONS

CLÉMENT FRAPPIER AND PATRICK OLIVIER

ABSTRACT. An exact quadrature formula for entire functions of exponential type is obtained. The nodes of the formula are zeros of the Bessel function of the first kind of order α. It generalizes and refines a known quadrature formula related to the sampling theorem. The uniqueness of the nodes is studied.

1. INTRODUCTION

Given any polynomial p of degree $< n$ and n distinct numbers $x_1, x_2, \ldots, x_n$, the classical Lagrange interpolation formula is

$$p(x) = \sum_{k=1}^{n} p(x_k) \frac{l(x)}{(x - x_k)\, l'(x_k)}, \tag{1}$$

where $l(x) = \prod_{j=1}^{n}(x - x_j)$. Multiplying both members of (1) by a weight function $w(x)$ and integrating, we obtain the quadrature formula

$$\int_{-1}^{1} w(x)\, p(x)\, dx = \sum_{k=1}^{n} \lambda_k\, p(x_k), \tag{2}$$

where

$$\lambda_k = \int_{-1}^{1} w(x) \frac{l(x)}{(x - x_k)\, l'(x_k)}\, dx.$$

In order to obtain a quadrature formula valid for all polynomials of degree $< 2n$, we consider the Hermite interpolation formula

$$p(x) = \sum_{k=1}^{n} (p(x_k)\, l_{k,0}(x) + p'(x_k)\, l_{k,1}(x)), \tag{3}$$

with

$$l_{k,0}(x) = \left(1 - \frac{l''(x_k)}{l'(x_k)}(x - x_k)\right)\left(\frac{l(x)}{(x - x_k)\, l'(x_k)}\right)^2$$

and

Received by the editor November 12, 1991.
1991 *Mathematics Subject Classification*. Primary 41A55, 33C10, 41A05.
Key words and phrases. Quadrature formula, nodes, entire functions, Bessel functions.
This research was supported by the Natural Sciences and Engineering Research Council of Canada Grant No. OGP0009331.

$$l_{k,1}(x) = (x - x_k)\left(\frac{l(x)}{(x-x_k)\, l'(x_k)}\right)^2 .$$

This time we obtain

$$\int_{-1}^{1} w(x)\, p(x) = \sum_{k=1}^{n} (\lambda_{k,0}\, p(x_k) + \lambda_{k,1}\, p'(x_k)) , \tag{4}$$

where

$$\lambda_{k,r} = \int_{-1}^{1} w(x)\, l_{k,r}(x)\, dx .$$

If we take, in particular, the nth orthogonal polynomial $l(x) = Q_n(x)$ associated with the weight function $w(x)$, then [6] we have $\lambda_{k,1} = 0$ and $\lambda_{k,0} = \lambda_k$, $1 \le k \le n$. Therefore, the quadrature formula (2) is valid for all polynomials of degree $< 2n$. Moreover, the zeros of $Q_n(x)$ are the only points having this property.

The Jacobi polynomial of degree n, $Q_n(x) = P_n^{(\alpha,\beta)}(x)$, is, with $w(x) = (1-x)^\alpha(1+x)^\beta$, an important special case . For the Chebyshev polynomial of the first kind of degree n,

$$T_n(x) = \frac{2^{2n}\,(n!)^2}{(2n)!}\, P_n^{(-1/2,-1/2)}(x) = \cos(n \arccos x) ,$$

we get the Gauss-Jacobi type formula

$$\int_{-1}^{1} \frac{p(x)}{\sqrt{1-x^2}}\, dx = \frac{\pi}{n} \sum_{k=1}^{n} p\left(\cos\frac{(2k-1)\pi}{2n}\right) . \tag{5}$$

This formula has been extended to entire functions. Let B_σ be the class of entire functions of exponential type σ, bounded on the real axis. If $f \in B_{2\tau}$ satisfies $f(x) = O\left(|x|^{-\delta}\right)$, $\delta > 1$, then [1]

$$\int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi}{\tau} \sum_{k=-\infty}^{\infty} f\left(\frac{(2k-1)\pi}{2\tau}\right) . \tag{6}$$

This latter is in fact valid under weaker integrability conditions [2]. Here the nodes $(2k-1)\pi/(2\tau)$, the zeros of $\cos\tau z$, are related to those of formula (5) by

$$\lim_{n\to\infty} \sqrt{\pi}\, \frac{P_n^{(-1/2,-1/2)}(\cos(\tau z/n))}{n^{-1/2}} = \cos\tau z .$$

In general [6, p.167],

$$\lim_{n\to\infty} \frac{P_n^{(\alpha,\beta)}(\cos(\tau z/n))}{n^\alpha} = \left(\frac{2}{\tau z}\right)^\alpha J_\alpha(\tau z) , \tag{7}$$

where $J_\alpha(z)$ is the Bessel function of the first kind of order α. Thus, the nodes in (6) are the zeros of $J_{-1/2}(\tau z)/(\tau z)^{-1/2}$; it appears natural to generalize (6) using as nodes the zeros of $J_\alpha(\tau z)/(\tau z)^\alpha$.

For the various properties of Bessel functions used in this paper, we refer the reader to [7].

2. Statement of the results

The function

$$\frac{J_\alpha(\tau z)}{(\tau z)^\alpha} = \sum_{k=0}^{\infty}(-1)^k \frac{(\tau z)^{2k}}{2^{\alpha+2k}\, k!\,\Gamma(k+\alpha+1)}$$

is an even entire function of exponential type τ. Let $j_k = j_k(\alpha)$, $k = \pm 1$, ± 2, ..., be the zeros of $J_\alpha(z)/z^\alpha$ ordered such that $j_{-k} = -j_k$ and $0 < |j_1| \leq |j_2| \leq \dots$. We are now ready to state our main result.

Theorem 1. *Let $\Re(\alpha) > -1$. For all $f \in B_{2\tau}$ such that $f(x) = O\left(|x|^{-\delta}\right)$, $x \to \pm\infty$, with $\delta > 2\Re(\alpha) + 2$, we have*

$$\int_0^\infty x^{2\alpha+1}\,(f(x) + f(-x))\,dx = \frac{2}{\tau^{2\alpha+2}} \sum_{k=1}^{\infty} \frac{j_k^{2\alpha}}{(J_\alpha'(j_k))^2} \left(f\left(\frac{j_k}{\tau}\right) + f\left(-\frac{j_k}{\tau}\right)\right). \tag{8}$$

The particular case $\alpha = -1/2$ leads us to formula (6). Indeed, we have

$$\frac{J_{-1/2}(\tau z)}{(\tau z)^{-1/2}} = \sqrt{\frac{2}{\pi}}\cos\tau z \qquad \text{and} \qquad j_k(-1/2) = (2k-1)\frac{\pi}{2}.$$

When $\alpha = 1/2$ we have

$$\frac{J_{1/2}(\tau z)}{(\tau z)^{1/2}} = \sqrt{\frac{2}{\pi}}\,\frac{\sin\tau z}{\tau z}$$

and the formula (8) becomes

$$\int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{\pi}{\tau} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{k\pi}{\tau}\right)^2 f\left(\frac{k\pi}{\tau}\right), \tag{9}$$

where $f(x) = O\left(|x|^{-\delta}\right)$, $x \to \pm\infty$, $\delta > 3$. Replacing $f(x)$ by

$$\frac{1}{x^2}\left(f(x) - f(0)\left(\frac{\sin\tau x}{\tau x}\right)^2\right),$$

where $f(x)$ can be supposed to be even, we obtain

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{\tau} \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\tau}\right), \tag{10}$$

where $f(x) = O\left(|x|^{-\delta}\right)$, $x \to \pm\infty$, $\delta > 1$. Up to a translation, this formula is equivalent to (6).

By considering a function of the form

$$f(z) = F(z)\,\frac{J_\alpha(\tau z)}{z^\alpha\,(\tau z - j_m)},$$

we obtain the following corollary of Theorem 1.

Corollary 1. *Let $\Re(\alpha) > -1$. For all $F \in B_\tau$ such that $F(x) = O\left(|x|^{-\delta}\right)$, $x \to \pm\infty$, with $\delta > \Re(\alpha) + 1/2$, we have*

$$F\left(\frac{j_m}{\tau}\right) = \frac{\tau^{\alpha+2}}{2j_m^\alpha} J'_\alpha(j_m) \int_0^\infty x^{\alpha+1} J_\alpha(\tau x) \left(\frac{F(x)}{\tau x - j_m} - \frac{F(-x)}{\tau x + j_m}\right) dx. \tag{11}$$

Taking in (11) the function

$$F(z) = \frac{J_\alpha(\tau z)}{z^\alpha (\tau z - j_l)},$$

we deduce the following property of Bessel functions.

Corollary 2. *Let $\Re(\alpha) > -1$. If m and l are integers, then*

$$\begin{aligned} &\frac{1}{2}\int_0^\infty x J_\alpha^2(x) \left(\frac{1}{(x - j_m)(x - j_l)} + \frac{1}{(x + j_m)(x + j_l)}\right) dx \\ &= \begin{cases} 0, & l \neq m, \\ 1, & l = m. \end{cases} \end{aligned} \tag{12}$$

Note that the case $l = m$ of (12) will be used in the proof of Theorem 1. So, we shall need an independent proof (see Lemma 4).

When $\alpha = 1/2$, in Corollary 2, we readily obtain the orthogonality property [4] of the family

$$\left\{\frac{\sin \pi(x - m)}{\pi(x - m)} : m \in \mathbb{Z}\right\},$$

namely,

$$\int_{-\infty}^\infty \frac{\sin \pi(x - m)}{\pi(x - m)} \frac{\sin \pi(x - l)}{\pi(x - l)} dx = \begin{cases} 0, & l \neq m, \\ 1, & l = m. \end{cases} \tag{13}$$

It has been proved [5] that formula (10) is the unique quadrature formula of the form

$$\int_{-\infty}^\infty f(x)\, dx = \sum_{k=-\infty}^\infty \lambda_k f(x_k), \tag{14}$$

which is valid for all $f \in B_{2\tau}$. The nodes x_k in (14) are completely characterized by an extremal problem: they are the roots of the function $\sin \tau x$, which minimizes the integral

$$\int_{-\infty}^\infty \left(\frac{s(x)}{x}\right)^2 dx,$$

over a certain subclass of B_τ. The following theorem is the corresponding ($\alpha = -1/2$) uniqueness result for quadrature formulas of the form

$$\int_0^\infty x^{2\alpha+1} (f(x) + f(-x))\, dx = \sum_{k=-\infty}^\infty \lambda_k f(x_k). \tag{15}$$

Observe that if α is not real then the uniqueness does not hold. The formula (8), when α is replaced by $\overline{\alpha}$, is still valid for the same class of functions. To see this we need only to replace $f(z)$ by $\overline{f(\overline{z})}$ and to observe that $\overline{j_k(\alpha)} = j_k(\overline{\alpha})$. Therefore, we assume that the sequence x_k, $-\infty < k < \infty$, in (15), is a sequence of distinct real numbers without an accumulation point in $\mathbb{R}$. We assume also that one of the nodes is a zero of $J_\alpha(\tau z)$. We associate, with every quadrature formula of the form (15) and every zero j_m of $J_\alpha(z)$ (which has only real zeros for $\alpha > -1$), a class of entire functions ω with the following properties:

(i) $\omega \in B_\tau$,
(ii) $\omega(x) \in \mathbb{R}$ for $x \in \mathbb{R}$,
(iii) $\omega(x) = O\left(|x|^{-\alpha-3/2}\right)$, $x \to \pm\infty$,
(iv) $\omega(x_k) = 0$, $k \neq m$,
(v) $\omega(j_m/\tau) = 1$.

Here the nodes x_k, $-\infty < k < \infty$, have been ordered such that $x_m = j_m/\tau$. We call every such function ω a nodal function.

Theorem 2. *Let* $\alpha > -1$. *Among all the quadrature formulas of the form* (15) *having* $\omega(x)$ *as nodal function, only one is valid for all* $f \in B_{2\tau}$ *satisfying* $f(x) = O\left(|x|^{-\delta}\right)$, $x \to \pm\infty$, $\delta > 2\alpha+2$. *This formula is* (8), *and the associated nodal function is*

$$\omega_e(x) := \frac{j_m^\alpha}{J_\alpha'(j_m)} \frac{J_\alpha(\tau x)}{x^\alpha\,(\tau x - j_m)}.$$

Moreover, ω_e *minimizes the integral*

$$\int_0^\infty x^{2\alpha+1}(\omega^2(x) + \omega^2(-x))\,dx,$$

over all nodal functions ω.

3. Auxiliary results

For every entire function of exponential type satisfying certain conditions at infinity, we have the classical sampling theorem, namely

$$f(x) = \sum_{k=-\infty}^{\infty} (-1)^k f\left(\frac{k\pi}{\tau}\right) \frac{\sin \tau x}{\tau x - k\pi}. \tag{16}$$

Integrating formally both members of (16), we obtain formula (10) but only for $f \in B_\tau$. In order to prove a result valid for $f \in B_{2\tau}$, we use an extension, to entire functions of exponential type, of formula (3). This extension (see Lemma 3, below) is a generalization and a refinement of (16).

From now on we may suppose $\tau = 1$. We need first a technical result.

Lemma 1. *Let* $z = Re^{i\theta}$ *be a complex number on the circle* $|z| = R := N\pi + \Re(\alpha)\pi/2 + \pi/4$, *where* N *is a large positive integer. There exists a positive constant* $C(\alpha)$ *such that*

$$|J_\alpha(z)| > \frac{C(\alpha)}{\sqrt{R}}\, e^{R|\sin\theta|}, \qquad |\theta| \le \pi. \tag{17}$$

Proof. Obviously, we may suppose $\Re(z) \ge 0$. In view of the asymptotic expansion

$$J_\alpha(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)(1 + O(|z|^{-1})), \tag{18}$$

it suffices to prove that

$$\left|\cos\left(z - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)\right| > K(\alpha)\, e^{R|\sin\theta|}. \tag{19}$$

Let $x = R\cos\theta$, $y = R\sin\theta$, $u = x - \Re(\alpha)\pi/2 - \pi/4$ and $v = y - \Im(\alpha)\pi/2$. Note that

$$\tag{20} \begin{aligned} &\cos\left(z - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right) = \cos(u + iv) \qquad \text{and} \\ &|\cos(u+iv)|^2 = \cos^2 u + \sinh^2 v. \end{aligned}$$

Let $R_k := k\pi + \Re(\alpha)\pi/2 + \pi/4$, $x_k := R_k - \pi/2$ and $y_0 := \Im(\alpha)\pi/2$. For $1 \le k \le N$, consider the two arcs of the circle $|z| = R$ inside the strip $x_k - \pi/2 \le x \le x_k + \pi/2$. Suppose that $z = x + iy$ is on an arc for which $x_k - \pi/2 \le x \le x_k + \pi/2$ and $|y - y_0| \ge \pi/4$. Using (20), we have

$$\left|\cos\left(z - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)\right|^2 = \cos^2 u + \sinh^2 v \ge \sinh^2 v \ge \left(\frac{1 - e^{-\pi/2}}{2}\right)^2 e^{2|v|},$$

whence

$$\tag{21} \left|\cos\left(z - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)\right| > C_1(\alpha)\, e^{R|\sin\theta|}.$$

If $z = x + iy$ is on an arc for which $x_k - \pi/2 \le x \le x_k + \pi/2$ and $|y - y_0| < \pi/4$, then $|x - x_k| > \sqrt{\pi^2/16 - v^2}$ (recall that N is taken sufficiently large). But $|x - x_k| = |u - k\pi + \pi/2| \le \pi/2$, and so

$$\begin{aligned} \left|\cos\left(z - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)\right|^2 &= \cos^2 u + \sinh^2 v = \sin^2\left(u - k\pi + \frac{\pi}{2}\right) + \sinh^2 v \\ &= \sin^2|x - x_k| + \sinh^2 v > \sin^2\left(\sqrt{\frac{\pi^2}{16} - v^2}\right) + \sinh^2 v =: A(v). \end{aligned}$$

The function $e^{-2|v|}A(v)$ is a positive continuous function in the closed interval $[-\pi/4, \pi/4]$. Thus, it has a positive minimum in that interval, say D. It follows that

$$\tag{22} \left|\cos\left(z - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)\right| > \sqrt{D}\, e^{|v|} > C_2(\alpha)\, e^{R|\sin\theta|}. \quad \square$$

We shall also need the following result, which is a simple variant of [3, Lemmas 1 and 2].

Lemma 2. *Let $f \in B_\sigma$ such that $f(x) = O\left(|x|^{-\delta}\right)$, $x \to \pm\infty$. If $R \to \infty$, then we have, uniformly for $|\theta| \le \pi$,*

$$\tag{23} f(Re^{i\theta}) = O\left(\frac{e^{\sigma R|\sin\theta|}}{R^\delta}\right).$$

Moreover,

$$\tag{24} f'(x) = O(|x|^{-\delta}), \qquad x \to \pm\infty.$$

Lemma 3. *Let $\Re(\alpha) > -1$. For all $f \in B_2$ such that $f(x) = O\left(|x|^{-\delta}\right)$, $x \to \pm\infty$, with $\delta > 2\Re(\alpha) + 1$, we have*

$$\tag{25} f(x) = \sum_{\substack{k=-\infty \\ k \ne 0}}^{\infty} [A_k(x) f(j_k) + B_k(x) f'(j_k)],$$

where

$$A_k(x) := \left(1 + \frac{2\alpha+1}{j_k}(x - j_k)\right)\left(\frac{j_k^\alpha J_\alpha(x)}{x^\alpha J'_\alpha(j_k)(x-j_k)}\right)^2 \tag{26}$$

and

$$B_k(x) := (x - j_k)\left(\frac{j_k^\alpha J_\alpha(x)}{x^\alpha J'_\alpha(j_k)(x-j_k)}\right)^2. \tag{27}$$

Proof. Consider the integral

$$I_N(x) := \frac{1}{2\pi i}\oint_{C_N} \frac{z^{2\alpha} f(z)}{(z-x) J_\alpha^2(z)}\, dz, \tag{28}$$

where C_N is the circle $|z| = R := N\pi + \Re(\alpha)\,\pi/2 + \pi/4$. As remarked before, the function $J_\alpha(z)/z^\alpha$ is entire. Using Lemmas 1 and 2, we have, for large R,

$$|I_N(x)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^{2\Re(\alpha)+1}\,|f(Re^{i\theta})|}{|Re^{i\theta} - x|\,|J_\alpha(Re^{i\theta})|^2}\, d\theta \le K(\alpha, x)\, R^{2\Re(\alpha)+1-\delta}.$$

Thus,

$$\lim_{N\to\infty} I_N(x) = 0. \tag{29}$$

On the other hand, using the residue theorem, we have

$$I_N(x) = \operatorname{Res}(z = x) + \sum_{|j_k|<R} \operatorname{Res}(z = j_k), \tag{30}$$

and the result follows after a few calculations by letting $N \to \infty$. □

The following properties of Bessel functions will be used, in conjunction with Lemma 3, to prove (8).

Lemma 4. *Let* $\Re(\alpha) > -1$. *For any zero* j_k *of* $J_\alpha(z)/z^\alpha$, *we have*

$$\int_0^\infty x\,\frac{J_\alpha^2(x)}{x^2 - j_k^2}\, dx = 0 \tag{31}$$

and

$$\int_0^\infty x\, J_\alpha^2(x)\left(\frac{1}{(x-j_k)^2} + \frac{1}{(x+j_k)^2}\right) dx = 2. \tag{32}$$

Proof. Let $R > \varepsilon > 0$. First we prove (31). We consider the curve Γ which is the union of the two intervals $[-R, -\varepsilon]$, $[\varepsilon, R]$ and the two semicircles C_ε, C_R, where $C_\delta := \{z\colon |z| = \delta \text{ and } \Im(z) \ge 0\}$. We have

$$\oint_\Gamma \frac{z\, J_\alpha(z)\, H_\alpha^{(1)}(z)}{z^2 - j_k^2}\, dz = 0, \tag{33}$$

where

$$H_\alpha^{(1)}(z) = \frac{J_{-\alpha}(z) - e^{-\alpha\pi i} J_\alpha(z)}{i \sin \alpha\pi}$$

is a Bessel function of the third kind, with the usual interpretation when α is an integer. From (33), we obtain

$$\int_{-R}^{-\varepsilon} F(x)\, dx + \int_{C_\varepsilon} F(z)\, dz + \int_\varepsilon^R F(x)\, dx + \int_{C_R} F(z)\, dz = 0, \tag{34}$$

where

$$F(z) := \frac{z\, J_\alpha(z)\, H_\alpha^{(1)}(z)}{z^2 - j_k^2}.$$

On C_ε we have $|z| = \varepsilon$; since $\lim_{z\to 0} J_\alpha(z)/z^\alpha = 1/\bigl(2^\alpha\,\Gamma(\alpha+1)\bigr)$, we obtain, if α is not an integer,

$$|F(z)| \le K_1(\alpha)\,\varepsilon + K_2(\alpha)\,\varepsilon^{1+2\Re(\alpha)}.$$

When $\alpha = n$ is an integer, we need to use, in addition, a representation formula for $H_\alpha^{(1)}(z)$; we have $H_n^{(1)}(z) = J_n(z) + i\,Y_n(z)$, with

$$\begin{aligned}\pi\, Y_n(z) = 2\left(\gamma + \ln\frac{z}{2}\right) J_n(z) - \sum_{\kappa=0}^{n-1} \frac{(n-\kappa-1)!}{\kappa!}\left(\frac{z}{2}\right)^{2\kappa-n}\\ - \sum_{\kappa=0}^{\infty}\frac{(-1)^\kappa}{\kappa!\,(n+\kappa)!}\left(\frac{z}{2}\right)^{2\kappa+n}\\ \times\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{\kappa}+\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n+\kappa}\right).\end{aligned} \tag{35}$$

We obtain

$$|F(z)| \le K_3(\alpha)\,\varepsilon + K_4(\alpha)\,\varepsilon^{1+2\Re(\alpha)}\,|\ln\varepsilon|.$$

It follows that

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} F(z)\,dz = 0 \quad \text{for } \Re(\alpha) > -1. \tag{36}$$

On C_R we use the asymptotic formula (18) and

$$H_\alpha^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{i\left(z-\alpha\frac{\pi}{2}-\frac{\pi}{4}\right)}(1 + O(|z|^{-1}))$$

to obtain

$$|F(z)| \le \frac{K_5(\alpha)}{R^2}.$$

Thus,

$$\lim_{R\to\infty}\int_{C_R} F(z)\,dz = 0. \tag{37}$$

Also, using $H_\alpha^{(1)}(x) - e^{\alpha\pi i}\,H_\alpha^{(1)}(-x) = 2J_\alpha(x)$, we get

$$\int_{-R}^{-\varepsilon} F(x)\,dx + \int_\varepsilon^R F(x)\,dx = \int_\varepsilon^R (F(x) + F(-x))\,dx = 2\int_\varepsilon^R x\,\frac{J_\alpha^2(x)}{x^2 - j_k^2}\,dx,$$

and so (31) follows from (34), (36), and (37).

Now we prove (32). We need to distinguish two cases according as j_k is real or not. When j_k is not real, we consider the contour Γ defined in the proof of (31). Only one zero, j_k or $-j_k$, is inside Γ. Thus, by the residue theorem,

$$\begin{aligned}&\frac{1}{2\pi i}\oint_\Gamma z\, J_\alpha(z)\, H_\alpha^{(1)}(z)\left(\frac{1}{(z-j_k)^2} + \frac{1}{(z+j_k)^2}\right) dz\\ &\qquad = j_k\, J_\alpha'(j_k)\, H_\alpha^{(1)}(j_k) = j_k\, J_\alpha'(j_k)\,\frac{J_{-\alpha}(j_k)}{i\sin\alpha\pi} = \frac{2}{\pi i},\end{aligned}$$

since

$$J'_\alpha(z)\,J_{-\alpha}(z) - J'_{-\alpha}(z)\,J_\alpha(z) = \frac{2\sin\alpha\pi}{\pi z}. \tag{38}$$

The integrals along C_ε and C_R tend to zero, as $\varepsilon \to 0$ and $R \to \infty$; the argument is analogous to the one used to obtain (36) and (37). Hence, using a decomposition of the form (34), we get

$$\frac{1}{2\pi i}\int_0^\infty x(J_\alpha(x)\,H_\alpha^{(1)}(x) - J_\alpha(-x)\,H_\alpha^{(1)}(-x))\left(\frac{1}{(x-j_k)^2} + \frac{1}{(x+j_k)^2}\right)dx = \frac{2}{\pi i},$$

which readily gives us (32).

Now we suppose that j_k is real. Consider the curve Γ_1 obtained from Γ by replacing the interval $[-j_k-\delta, -j_k+\delta]$ by the semicircle $C_{1,\delta} = \{z : |z+j_k| = \delta \text{ and } \Im(z) \geq 0\}$, where δ is sufficiently small. We have

$$\oint_{\Gamma_1} F_1(z)\,dz = 0, \quad \text{where } F_1(z) := \frac{z\,J_\alpha(z)\,H_\alpha^{(1)}(z)}{(z+j_k)^2}. \tag{39}$$

It follows that

$$\int_{-\infty}^{-j_k-\delta} F_1(x)\,dx + \int_{C_{1,\delta}} F_1(z)\,dz + \int_{-j_k+\delta}^{0} F_1(x)\,dx + \int_0^\infty F_1(x)\,dx = 0. \tag{40}$$

Similarly,

$$\int_{-\infty}^{0} F_2(x)\,dx + \int_0^{j_k-\delta} F_2(x)\,dx + \int_{C_{2,\delta}} F_2(z)\,dz + \int_{j_k+\delta}^\infty F_2(x)\,dx = 0, \tag{41}$$

where

$$F_2(z) := \frac{z\,J_\alpha(z)\,H_\alpha^{(1)}(z)}{(z-j_k)^2} \quad \text{and} \quad C_{2,\delta} = \{z : |z-j_k| = \delta \text{ and } \Im(z) \geq 0\}.$$

We add (40) and (41) to obtain

$$\begin{aligned} 0 = &\int_0^{j_k-\delta} (F_1(-x)+F_2(x))\,dx + \int_{j_k+\delta}^\infty (F_1(-x)+F_2(x))\,dx \\ &+ \int_0^\infty (F_1(x)+F_2(-x))\,dx + \int_{C_{1,\delta}} F_1(z)\,dz + \int_{C_{2,\delta}} F_2(z)\,dz, \end{aligned} \tag{42}$$

whence

$$\begin{aligned} 0 = 2&\int_0^{j_k-\delta} x\,\frac{J_\alpha^2(x)}{(x-j_k)^2}\,dx + 2\int_{j_k+\delta}^\infty x\,\frac{J_\alpha^2(x)}{(x-j_k)^2}\,dx \\ &+ 2\int_0^\infty x\,\frac{J_\alpha^2(x)}{(x+j_k)^2}\,dx + \int_{C_{1,\delta}} F_1(z)\,dz + \int_{C_{2,\delta}} F_2(z)\,dz. \end{aligned} \tag{43}$$

Taking the limit as $\delta \to 0$, we get

$$\begin{aligned} 0 = 2&\int_0^\infty x\,J_\alpha^2(x)\left(\frac{1}{(x-j_k)^2} + \frac{1}{(x+j_k)^2}\right)dx \\ &+ \lim_{\delta\to 0}\left(\int_{C_{1,\delta}} F_1(z)\,dz + \int_{C_{2,\delta}} F_2(z)\,dz\right). \end{aligned} \tag{44}$$

It remains to evaluate the last limit. We have, if α is not an integer,

$$\begin{aligned}\lim_{\delta\to 0}&\left(\int_{C_{1,\delta}} F_1(z)\,dz\right)\\ &= i\lim_{\delta\to 0}\int_{\pi}^{0}(-j_k+\delta e^{i\theta})\frac{J_\alpha(-j_k+\delta e^{i\theta})}{\delta e^{i\theta}}H_\alpha^{(1)}(-j_k+\delta e^{i\theta})\,d\theta\\ &= i\pi\, j_k\, J'_\alpha(-j_k)\,H_\alpha^{(1)}(-j_k) = \pi\, j_k\,\frac{J'_\alpha(-j_k)\,J_{-\alpha}(-j_k)}{\sin\alpha\pi}\\ &= -\pi\, j_k\,\frac{J'_\alpha(j_k)\,J_{-\alpha}(j_k)}{\sin\alpha\pi}\end{aligned}$$

and

$$\lim_{\delta\to 0}\left(\int_{C_{2,\delta}} F_2(z)\,dz\right) = -\pi\, j_k\,\frac{J'_\alpha(j_k)\,J_{-\alpha}(j_k)}{\sin\alpha\pi}.$$

Using (38), we readily obtain

$$\lim_{\delta\to 0}\left(\int_{C_{1,\delta}} F_1(z)\,dz + \int_{C_{2,\delta}} F_2(z)\,dz\right) = -4,$$

and (32) follows from (44). When α is an integer, the function $H_\alpha^{(1)}(z)$ has to be replaced by its limit. □

4. Proofs of the theorems

We may obviously suppose $\tau = 1$.

Proof of Theorem 1. Using Lemma 3, we have

$$f(x)+f(-x) = \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} (A_k(x)+A_k(-x))f(j_k) + (B_k(x)+B_k(-x))f'(j_k)\,,$$

that is,

$$\begin{aligned}f(x)+f(-x) &= \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \frac{j_k^{2\alpha}}{(J'_\alpha(j_k))^2}\,\frac{J_\alpha^2(x)}{x^{2\alpha}}\left(\frac{1}{(x-j_k)^2}+\frac{1}{(x+j_k)^2}\right)f(j_k)\\ &\quad+\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} 2\,(2\alpha+1)\,\frac{j_k^{2\alpha}}{(J'_\alpha(j_k))^2}\,\frac{J_\alpha^2(x)}{x^{2\alpha}\,(x^2-j_k^2)}\,f(j_k)\\ &\quad+\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} 2\,\frac{j_k^{2\alpha+1}}{(J'_\alpha(j_k))^2}\,\frac{J_\alpha^2(x)}{x^{2\alpha}\,(x^2-j_k^2)}\,f'(j_k)\,.\end{aligned}\tag{45}$$

Multiplying by $x^{2\alpha+1}$ and integrating, we obtain formally the formula (8) with the help of Lemma 4. Hence, it remains to justify the interchanges of the order of integration and summation. First observe that, in order to prove formula (8), we may assume

$$f(x) = O(|x|^{-\delta}), \qquad x\to\pm\infty,\ \delta > 2\Re(\alpha)+6.\tag{46}$$

Indeed, if $f(x) = O\left(|x|^{-\delta}\right)$, $x \to \pm\infty$, with only $\delta > 2\Re(\alpha) + 2$, then the functions

$$g_\varepsilon(z) := \left(\frac{\sin \varepsilon z}{\varepsilon z}\right)^4 f(z)\,, \qquad \varepsilon > 0\,, \quad f \in B_{2\tau}\,,$$

satisfy the hypothesis of Theorem 1 with τ replaced by $\tau+2\varepsilon$ and $\delta > 2\Re(\alpha)+2$ replaced by $\delta > 2\Re(\alpha) + 6$. Thus,

$$\begin{aligned} &\int_0^\infty x^{2\alpha+1}\left(g_\varepsilon(x) + g_\varepsilon(-x)\right)dx \\ &\quad = \frac{2}{(\tau+2\varepsilon)^{2\alpha+2}} \sum_{k=1}^{\infty} \frac{j_k^{2\alpha}}{(J'_\alpha(j_k))^2}\left(g_\varepsilon\left(\frac{j_k}{\tau+2\varepsilon}\right) + g_\varepsilon\left(-\frac{j_k}{\tau+2\varepsilon}\right)\right). \end{aligned} \tag{47}$$

The passage to the limit as $\varepsilon \to 0$, in (47), is easily justified using

$$\left|\frac{\sin \varepsilon x}{\varepsilon x}\right| \le 1 \qquad \text{and} \qquad f\left(\pm\frac{j_k}{\tau+2\varepsilon}\right) = O\left(\frac{(\tau+2\varepsilon)^\delta}{|j_k|^\delta}\right).$$

Now we must prove, in particular, that

$$\begin{aligned} &\int_0^\infty \left(\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} 2\,\frac{j_k^{2\alpha+1}}{(J'_\alpha(j_k))^2}\,\frac{x\,J_\alpha^2(x)}{x^2-j_k^2}\,f'(j_k)\right)dx \\ &\quad = \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} 2\,\frac{j_k^{2\alpha+1}}{(J'_\alpha(j_k))^2}\,f'(j_k)\int_0^\infty \frac{x\,J_\alpha^2(x)}{x^2-j_k^2}\,dx. \end{aligned} \tag{48}$$

The other interchanges could be proved, with a slight simplification, as is done in the following argument. In view of Lebesgue's dominated convergence theorem, it suffices to show that there exists an integrable function $G(x)$ such that

$$\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \lambda_k(x) \le G(x)\,, \quad \text{where } \lambda_k(x) := \left|\frac{j_k^{2\alpha+1}}{(J'_\alpha(j_k))^2}\,\frac{x\,J_\alpha^2(x)}{x^2-j_k^2}\,f'(j_k)\right|. \tag{49}$$

For small values of x, we use $J_\alpha(x) = O\left(|x|^{\Re(\alpha)}\right)$ as $x \to 0$, (24) and the hypothesis (46) to obtain

$$\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \lambda_k(x) \le K_1(\alpha)\,x^{2\Re(\alpha)+1}.$$

For large values of x, we write

$$\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \lambda_k(x) = I_1 + I_2 + I_3\,, \tag{50}$$

where

$$I_1 := \sum_{|j_k|<\frac{x}{2}} \lambda_k(x)\,, \quad I_2 := \sum_{\frac{x}{2}\le|j_k|\le\frac{3x}{2}} \lambda_k(x) \quad \text{and} \quad I_3 := \sum_{\frac{3x}{2}<|j_k|} \lambda_k(x).$$

We have, for $x/2 \le |j_k| \le 3x/2$,

$$\left|\frac{J_\alpha^2(x)}{x^2 - j_k^2}\right| \le C_1(\alpha)\,,$$

which follows from

$$J_\alpha(x) = \int_{\pm j_k}^{x} J'_\alpha(u)\,du$$

and the asymptotic formula

$$J'_\alpha(u) = -\sqrt{\frac{2}{\pi u}}\,\sin\left(u - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)(1 + O(|u|^{-1}))\,.$$

Thus, using also (46) and (24), we obtain

$$I_2 \le \frac{C_1(\alpha)}{x^2} \sum_{\frac{x}{2} \le |j_k| \le \frac{3x}{2}} |j_k|^{2\Re(\alpha)+2-\delta} x^3 \le 8\frac{C_1(\alpha)}{x^2} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |j_k|^{2\Re(\alpha)+5-\delta} \le \frac{C_2(\alpha)}{x^2}\,.$$

In order to obtain an upper bound for I_1 and I_3, we use (46), (24) and the above asymptotic formula for J'_α. We get

$$I_1 \le C_3(\alpha) \sum_{|j_k| < \frac{x}{2}} \frac{|j_k|^{2\Re(\alpha)+2-\delta}}{x^2 - |j_k|^2} \le \frac{4}{3}\frac{C_3(\alpha)}{x^2} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |j_k|^{2\Re(\alpha)+2-\delta} \le \frac{C_4(\alpha)}{x^2}$$

and

$$I_3 \le C_5(\alpha) \sum_{\frac{3x}{2} < |j_k|} \frac{|j_k|^{2\Re(\alpha)+2-\delta}}{|j_k|^2 - x^2} \le \frac{4}{5}\frac{C_5(\alpha)}{x^2} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |j_k|^{2\Re(\alpha)+2-\delta} \le \frac{C_6(\alpha)}{x^2}\,.$$

From (50), we deduce that

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \lambda_k(x) \le \frac{K_2(\alpha)}{x^2}\,,$$

for large values of x. □

Proof of Theorem 2. Consider two quadrature formulas of the form (15) with nodal functions $\omega_1(x)$ and $\omega_2(x)$, respectively. Suppose $\omega_1(x) \not\equiv \omega_2(x)$, and let $h(x) := \omega_1(x) - \omega_2(x)$. The function $h(x)\,\omega_1(x)$ is in B_2 and vanishes at the nodes x_k (including $k = m$) associated with the quadrature formula with $\omega_1(x)$ as nodal function. Moreover, $h(x)\,\omega_1(x) = O\left(|x|^{-2\alpha-3}\right)$, $x \to \pm\infty$. Therefore,

$$\int_0^\infty x^{2\alpha+1}\,(h(x)\,\omega_1(x) + h(-x)\,\omega_1(-x))\,dx = 0\,. \tag{51}$$

Since $h(x) \not\equiv 0$, we have

$$\int_0^\infty x^{2\alpha+1}\,(h^2(x) + h^2(-x))\,dx > 0\,,$$

and so, using (51),

$$\int_0^\infty x^{2\alpha+1}\,(\omega_2^2(x)+\omega_2^2(-x))\,dx > \int_0^\infty x^{2\alpha+1}\,(\omega_1^2(x)+\omega_1^2(-x))\,dx\,. \tag{52}$$

Considering, instead of $h(x)\,\omega_1(x)$, the function $h(x)\,\omega_2(x)$, we are led similarly to

$$\int_0^\infty x^{2\alpha+1}\,(\omega_1^2(x)+\omega_1^2(-x))\,dx > \int_0^\infty x^{2\alpha+1}\,(\omega_2^2(x)+\omega_2^2(-x))\,dx\,, \tag{53}$$

which contradicts (52). Thus $\omega_1(x)\equiv\omega_2(x)$.

Since $\omega_e(x)$ is the nodal function associated with (8), we deduce, by the same argument, that

$$\int_0^\infty x^{2\alpha+1}\,(\omega^2(x)+\omega^2(-x))\,dx > \int_0^\infty x^{2\alpha+1}\,(\omega_e^2(x)+\omega_e^2(-x))\,dx\,,$$

for all nodal functions $\omega(x)\not\equiv\omega_e(x)$. □

5. Remarks and examples

5.1. Theorem 1 does not remain valid for the class B_σ if $\sigma > 2\tau$. Any function of the form

$$f_*(z) = \frac{J_\alpha(\tau z)}{z^\alpha}\,\frac{J_{\overline{\alpha}}(\tau z)}{z^{\overline{\alpha}}}\left(\frac{J_{\eta/2}\left(\frac{\varepsilon z}{2}\right)}{z^{\eta/2}}\right)^2\,, \qquad \eta\,,\varepsilon>0\,,$$

is a counterexample. The function f_* is in the class $B_{2\tau+\varepsilon}$ and $f_*(x) = O\left(|x|^{-\delta}\right)$, $x\to\pm\infty$, with $\delta = 2\Re(\alpha)+2+\eta$. The summation in the right-hand member of (8) is clearly zero, but the integral of the left-hand member is positive, since

$$f_*(x) = \left|\frac{J_\alpha(\tau x)}{x^\alpha}\right|^2\left(\frac{J_{\eta/2}(\varepsilon x/2)}{x^{\eta/2}}\right)^2.$$

5.2. The hypothesis $\delta > 2\Re(\alpha)+2$ cannot be relaxed in Theorem 1. Consider the function

$$f_*(z) = \frac{J_\alpha(\tau z)\,J_{\alpha+1}(\tau z)}{z^{2\alpha+1}}\,.$$

This function is in the class $B_{2\tau}$ and $f_*(x) = O\left(|x|^{-\delta}\right)$, $x\to\pm\infty$, with $\delta = 2\Re(\alpha)+2$. The summation in the right-hand member of (8) is clearly zero but, for $\Re(\alpha) > -1$,

$$\int_0^\infty x^{2\alpha+1}\,(f_*(x)+f_*(-x))\,dx = 2\int_0^\infty J_\alpha(\tau x)\,J_{\alpha+1}(\tau x)\,dx = \frac{1}{\tau}\,.$$

5.3. When $\alpha = m+1/2$, where m is a nonnegative integer, the functions $J_\alpha(z)$ take the following explicit form:

$$J_{m+1/2}(z) = \sqrt{\frac{2z}{\pi}}\,z^m\left(-\frac{1}{z}\frac{d}{dz}\right)^m\left(\frac{\sin z}{z}\right).$$

In the case $m=1$, formula (8) becomes

$$\int_{-\infty}^\infty x^4\,f(x)\,dx = \frac{32\pi}{\tau^5}\sum_{k=-\infty}^{\infty} r_k^2\,(1+r_k^2)\,f\left(\frac{r_k}{\tau}\right)\,, \tag{54}$$

where $f\in B_{2\tau}$ satisfies $f(x) = O\left(|x|^{-\delta}\right)$, $x\to\pm\infty$, with $\delta>5$ and r_k, $-\infty<k<\infty$, are the roots of the equation $\tan x = x$.

5.4. Let

$$\varphi_m(z) := \frac{\sqrt{2}\, j_m\, J_\alpha(z)}{z^\alpha\,(z^2 - j_m^2)}.$$

If we take, in Theorem 1, $\tau = 1$ and $f(z) = \varphi_m(z)\,\varphi_l(z)$, then we obtain

$$\int_0^\infty x^{2\alpha+1}\,\varphi_m(x)\,\varphi_l(x)\,dx = \begin{cases} 0, & l \neq m, \\ 1, & l = m. \end{cases} \tag{55}$$

In other words, the family $\{\varphi_m : m = \pm 1, \pm 2, \dots\}$ is orthonormal on $[0, \infty)$ with weight function $x^{2\alpha+1}$.

Bibliography

1. R. P. Boas, Jr., *Summation formulas and band-limited signals*, Tôhoku Math. J. **24** (1972), 121–125.
2. C. Frappier et Q. I. Rahman, *Une formule de quadrature pour les fonctions entières de type exponentiel*, Ann. Sci. Math. Québec **10** (1986), 17–26.
3. R. Gervais, Q. I. Rahman and G. Schmeisser, *Representation and approximation of functions via* (0, 2)*-interpolation*, J. Approx. Theory **50** (1987), 89–110.
4. G. H. Hardy, *Notes on special systems of orthogonal functions* IV*: the orthogonal functions of Whittaker's cardinal series*, Proc. Cambridge Philos. Soc. **37** (1941), 331–348.
5. Q. I. Rahman and G. Schmeisser, *An analogue of Turán's quadrature formula*, Constr. Theory of Functions, Sofia, 1988, pp. 405–412.
6. G. Szegö, *Orthogonal polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ., Vol. 23, Providence, R.I., 1975.
7. G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed., Cambridge Univ. Press, Cambridge, 1952.

École Polytechnique, Département de Mathématiques Appliquées, CP 6079, Succ. A, Montréal (PQ), Canada, H3C 3A7